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ADDRESS

Delivered by the President, Mr. J. W. L. Glaisher, on presenting the Gold Medal of the Society to Mr. G. W. Hill.

The Council have awarded the Gold Medal to Mr. George William Hill for his researches upon the Lunar Theory; and it now becomes my duty to lay before you the grounds of this award.

The investigations of Mr. Hill's which the Council have had principally in view are contained in the memoir "On the Part of the Motion of the Lunar Perigee, which is a Function of the Mean Motions of the Sun and Moon," which was published by him ten years ago at Cambridge, Mass. The merits of Mr. Hill's treatment of this question are such, that, even if this memoir stood by itself as his sole contribution to physical astronomy, the Council would have felt themselves justified in recognising its value by the highest mark of appreciation which it is in their power to confer. Mr. Hill's object in this memoir is to determine with absolute accuracy the motion of the Moon's perigee, on the assumption that the Sun's orbit has no eccentricity, that the Moon's motion takes place in the plane of the ecliptic, and that the eccentricity of the Moon's orbit is indefinitely small: in other words, to determine an absolutely accurate value of that part of c which depends upon m alone.

It will be remembered that this quantity c, which expresses the mean motion of the Moon's perigee, has played an important part in the early history of the theory of gravitation. Newton proved a general proposition from which it follows that the disturbing action of the Sun necessarily produces a continual advance of the Moon's perigee; and he gave a numerical example which has been generally regarded as his calculation of the theoretical amount of this advance in the case of the Moon.[*] The mean monthly progression obtained in this example

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[*] Lib. I., Sect. ix., Prop. xlv., Cor. 2. The concluding words "Apsis lunæ est duplo velocior circiter" have been quoted in support of the view that the motion of the lunar apsides is the question considered in this corollary. These words were, however, intended to have exactly the opposite meaning. This can be shown by comparing the three editions of the *Principia*; further confirmation is also afforded by certain papers lately found by Professor Adams in the Portsmouth collection of Newton manuscripts.

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is 1° 31′ 28″, whereas the observed monthly progression of the Moon is about double this amount. This discrepancy formed one of the chief impediments to the acceptance of the Newtonian system, and an interval of sixty years elapsed before theory and observation were brought into agreement. This was effected by Clairaut, who was on the point of publishing a new hypothesis with respect to the law of gravitation, when it occurred to him to extend the approximation to the third order. He found that the numerical value of the new term was nearly as great as that of the term of the second order. The supposed Newtonian value of $1-c$ was $\frac{3}{4}m^2$, and Clairaut's additional term was $\frac{225}{32}m^3$; amounting to about $\frac{7}{10}$ of the previous term. Thus the circumstance which had for some time seemed almost fatal to the Newtonian theory became one of its strongest confirmations.

The series of which the first two terms were calculated by Newton and Clairaut has been continued by Delaunay as far as the term involving m^9. The progression of the terms is such as we might anticipate from the first two; the coefficients increase rapidly; and, though the smallness of m causes the terms to diminish, their rate of convergence is very slow.

The problem which Mr. Hill set before himself for solution was the exact determination of the numerical value of the quantity represented by this series. The development in powers of m, besides being extremely difficult and laborious to obtain, was from its nature ill adapted for the calculation of c with all the precision that could be desired. Mr. Hill accordingly had recourse to an entirely different method of obtaining the numerical value of this quantity.

Starting with the general equations of motion, he finds the differential equations which determine the inequalities having the first power of the lunar eccentricity as a factor. By a remarkable series of analytical transformations, the solution of these differential equations is made to depend upon that of a single differential equation of the peculiarly simple and elegant form

$$\frac{d^2\mathrm{w}}{d\tau^2}+\Theta\mathrm{w}=0,$$

where τ denotes the mean angular distance between the Sun and Moon, and Θ can be developed in a periodic series of the form

$$\Theta_0+\Theta_1\cos 2\tau+\Theta_2\cos 4\tau+\ \&\text{c.}$$

If Θ_1, Θ_2, &c. are to a considerable degree smaller than Θ_0, an approximate solution of the equation is

$$\mathrm{w}=\mathrm{K}e^{ic\tau}+\mathrm{K}'e^{-ic\tau},$$

where i denotes $\sqrt{-1}$, K and K′ are arbitrary constants, and c denotes the square root of Θ_0. When the additional terms are

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taken into account, the effect is to modify this value of c, and also to add to w new terms of the form

$$e^{\pm i(c+2p)\tau}.$$

We may, therefore, take as a particular integral

$$w = \Sigma_p b_p e^{i(c+2p)\tau};$$

and by substituting this value of w in the differential equation, we obtain an infinite system of algebraical equations, each consisting of an infinite number of terms. This system of equations suffices to determine the ratios of all the coefficients b_1, b_2, &c., to one of them, as b_0, and also the value of c, the ratio of the synodic to the anomalistic month.

If from these equations we eliminate all the b's, we obtain a symmetrical determinant of an infinite order involving c, which, when equated to zero, determines this quantity. This equation Mr. Hill writes

$$\mathfrak{D}(c) = 0.$$

A first approximation to the value of c is $\sqrt{\Theta_0}$; a second and much more accurate approximation is given by

$$c = \sqrt{1 + \sqrt{(\Theta_0 - 1)^2 - \Theta_1^2}}.$$

This very simple expression for an approximate value of the motion of the Moon's perigee is obtained by neglecting quantities of the order m^5 in the determinantal equation $\mathfrak{D}(c) = 0$. The numerical values of Θ_0 and Θ_1 are

$$\Theta_0 = \quad 1{\cdot}1588439$$
$$\Theta_1 = -0{\cdot}0570440,$$

and by substituting these values in the above expression for c we find

$$c = 1{\cdot}0715632$$

whence

$$1 - c = 1 - c(1 - m) = 0{\cdot}008591.$$

This is about $\frac{1}{60}$ in excess of the value 0·008452 given by observation; the difference being mainly due to the neglect of the inclination of the lunar orbit.*

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* Mr. Hill has also given the following curious formula, which is obtained by neglecting quantities of the order m^7: if

$$\tan\theta = \frac{\pi\Theta_1^2}{4\sqrt{\Theta_0}(\Theta_0 - 1)},$$

then c is given by the equation

$$\cos\pi c = \frac{\cos(\pi\sqrt{\Theta_0} - \theta)}{\cos\theta}.$$

This formula, though it involves the same coefficients Θ_0 and Θ_1 as the approximate formula given in the text, is much more exact. It gives $1 - c = 0{\cdot}00857210$, which is correct to the sixth decimal. See the reprint of Mr. Hill's paper in the *Acta Mathematica*, vol. viii. (1886), p. 28.

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In order to deduce a perfectly accurate value of c from the equation $\mathfrak{D}(c) = 0$, Mr. Hill proceeds to consider the nature of the roots of this equation. He points out that if it is satisfied by $c = c_0$ it must necessarily be satisfied also by all quantities of the form $\pm (c_0 + 2p)$, p being an integer; and he is thus led to put

$$\mathfrak{D}(c) = A\{\cos(\pi c) - \cos(\pi c_0)\}$$

The constant A is next determined, and the determinant $\mathfrak{D}(c)$ is replaced by a new determinant $\nabla(c)$, derived from it by the equation

$$\nabla(c) = \frac{1}{A}\mathfrak{D}(c)$$

so that

$$\nabla(c) = \cos(\pi c) - \cos(\pi c_0).$$

This equation is true identically. Putting $c = 0$, and subtracting, we thus find

$$\cos(\pi c) = 1 - \nabla(0),$$

as the subscript zero may now be omitted. This very remarkable equation is equivalent to $\mathfrak{D}(c) = 0$, and gives by its solution the value of c, which we are seeking; the particular root to be chosen being that to which $\sqrt{\Theta_0}$ is a first approximation.

Mr. Hill makes still one further change in order to replace $\nabla(0)$ by a determinant in which all the terms in the principal diagonal shall be unity. Denoting this new determinant by (0), the final form of the equation giving c is expressed in the form:

$$\frac{\sin^2(\frac{1}{2}\pi c)}{\sin^2(\frac{1}{2}\pi\sqrt{\Theta_0})} = \Box(0),$$

where $\Box(0)$ denotes the very elegant determinant

$$\left|\begin{array}{ccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\ldots, & 1, & -\frac{\Theta_1}{4^2-\Theta_0}, & -\frac{\Theta_2}{4^2-\Theta_0}, & -\frac{\Theta_3}{4^2-\Theta_0}, & -\frac{\Theta_4}{4^2-\Theta_0}, & \ldots \\
\ldots, & -\frac{\Theta_1}{2^2-\Theta_0}, & 1, & -\frac{\Theta_1}{2^2-\Theta_0}, & -\frac{\Theta_2}{2^2-\Theta_0}, & -\frac{\Theta_3}{2^2-\Theta_0}, & \ldots \\
\ldots, & -\frac{\Theta_2}{0^2-\Theta_0}, & -\frac{\Theta_1}{0^2-\Theta_0}, & 1, & -\frac{\Theta_1}{0^2-\Theta_0}, & -\frac{\Theta_2}{0^2-\Theta_0}, & \ldots \\
\ldots, & -\frac{\Theta_3}{2^2-\Theta_0}, & -\frac{\Theta_2}{2^2-\Theta_0}, & -\frac{\Theta_1}{2^2-\Theta_0}, & 1, & -\frac{\Theta_1}{2^2-\Theta_0}, & \ldots \\
\ldots, & -\frac{\Theta_4}{4^2-\Theta_0}, & -\frac{\Theta_3}{4^2-\Theta_0}, & -\frac{\Theta_2}{4^2-\Theta_0}, & -\frac{\Theta_1}{4^2-\Theta_0}, & 1, & \ldots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right|$$

In the lunar theory the quantity Θ_p is of the $2p^{\text{th}}$ order. Mr. Hill, with great analytical skill developes this determinant as

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far as terms of the twelfth order. It is with reluctance that I abstain from reproducing the remarkable expression which he obtains. It occupies eight lines, and consists of terms of the fourth order, the eighth order, and the twelfth order. It is exactly the same result as would be obtained if, starting with the equation $\mathfrak{D}(c) = 0$, and assuming $c = \sqrt{\Theta_0}$ as a first approximation, we expanded the function $\sin^2(\frac{1}{2}\pi c)$ in ascending powers and products of the coefficients Θ_1, Θ_2, &c.

Passing now to numbers, Mr. Hill puts

$$n = 17325594''\cdot 06085$$
$$n' = 1295977''\cdot 41516$$

and deduces the value of $\Theta_0, \Theta_1, \ldots \Theta_7$ to 15 places of decimals, and by substituting these values in his expression for $\square(0)$ he obtains finally

$$c = 1\cdot 07158\ 32774\ 16012.$$

whence

$$1 - c = 0\cdot 00857\ 25730\ 04864.$$

By a process of substitution in the original equations a verification of this value of c is afforded.

It is interesting to compare Mr. Hill's value with the value given by Delaunay's expression in powers of m, viz.:

$$1 - c = \frac{3}{4}m^2 + \frac{225}{32}m^3 + \frac{4071}{128}m^4 + \frac{265493}{2048}m^5 + \frac{12822631}{24576}m^6$$
$$+ \frac{1273925965}{589824}m^7 + \frac{71028685589}{7077888}m^8 + \frac{32145882707741}{679477248}m^9.$$

Using the values of n and n' given above, the numerical values of the terms of this series are

m^2	0·00419 6429	m^6	0·00009 1395
m^3	0·00294 2798	m^7	0·00002 8300
m^4	0·00099 5700	m^8	0·00000 9836
m^5	0·00030 3577	m^9	0·00000 3468

giving 0·00857 1503 as the value of $1 - c$.

By comparing this value with Mr. Hill's it appears that the sum of the remainder of Delaunay's series is 0·0000 1070, which is rather less than might be inferred by induction from the terms of the series itself. In spite of the labour expended by Delaunay in computing eight terms of this series, the result does not suffice to give correctly the first four significant figures of $1 - c$. On the other hand, in Mr. Hill's process, the highest terms included in the calculation of the expression for $\square(0)$ are only of the twelfth order, and yet they suffice to give c correctly to the fifteenth decimal. As well as can be judged by induction, it

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would be necessary, Mr. Hill thinks, to prolong Delaunay's series as far as m^{27} in order to obtain an equally precise result.

I am sensible that a presidential address does not in general afford a suitable opportunity for mathematical exposition; but I have felt that on the present occasion the novelty and intrinsic interest of the mathematical methods employed justify me in the attempt I have made to give a brief sketch of the remarkable investigation by which Mr. Hill has arrived at his numerical value of c. It will be observed that the problem of the determination of this quantity is attacked entirely *de novo*, from first principles, by a peculiar analytical method devised for this especial purpose. The object is to obtain the numerical value of that part of c which depends upon m only; and this object is attained with a degree of precision that sets the problem at rest for ever.

The mathematical process is very peculiar. Not only is it quite different from any of the methods with which the lunar theory is associated, but it even displays novelty from the point of view of the pure mathematician. I am not aware that actual use has ever been previously made of an infinite determinant in any of the applications of mathematics, or that the development of such a determinant (by proceeding outwards from its central constituent, as it were) has ever been the subject of mathematical investigation. One cannot admire too highly the courage and skill with which Mr. Hill has dealt with the new mathematical questions to which his methods have led him. Not only has he availed himself of the modern instruments which the advance of mathematics had placed within his reach, but he has successfully grappled with the novel mathematical difficulties which have accompanied their use. The calculation of the determinants formed from the 3^2, 5^2, 7^2, &c. constituents symmetrically disposed with respect to the central constituent of an infinite determinant, and the consideration of the convergency of these determinants, are questions of mathematical interest which are well deserving of study on their own account, apart from the practical applications in which, as Mr. Hill has shown, the results of such investigations are required.

Although the memoir of Mr. Hill's to which I have so far confined myself was the first in order of publication of his papers on the Lunar Theory, its true chronological place is subsequent to his "Researches on the Lunar Theory," which appeared in the first volume of the "American Journal of Mathematics" (1878). In the calculation of the quantities Θ_0, Θ_1, Θ_2, &c. (which depend only upon the case of no eccentricities), Mr. Hill derived his numerical data from an investigation of his own to be subsequently published. This investigation is contained in the "Researches," which may be regarded as introductory to the memoir relating to the motion of the perigee.

At the beginning of the "Researches" Mr. Hill divides the periodic developments of the lunar co-ordinates into classes of

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terms, and proposes to treat the following five classes of inequalities:

1. Those which depend only on the ratio of the mean motions of the Sun and Moon.
2. Those which are proportional to the lunar eccentricity.
3. Those which are proportional to the sine of the lunar inclination.
4. Those which are proportional to the solar eccentricity.
5. Those which are proportional to the solar parallax.

He also promises a general method by which these investigations may be extended so as to cover the whole ground of the Lunar Theory.

Only two chapters of the "Researches" were published: the first relates to the general differential equations of motion, and the second to the first of the five classes of inequalities mentioned above. The memoir on the motion of the perigee belongs to the second class; so that we are in possession of Mr. Hill's general methods and their application to two out of the five classes of inequalities.

Passing over Mr. Hill's able discussion of the general equations of motion, I proceed to give a brief sketch of the analytical method by which he treats the first two classes of inequalities.

If the axis of x rotate with uniform angular velocity n', so as always to pass through the Sun, the differential equations of motion in the ordinary form are

$$\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} + \left(\frac{\mu}{r^3} - 3n'^2\right)x = 0,$$

$$\frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} + \frac{\mu}{r^3}y \qquad = 0.$$

Mr. Hill transforms these equations as follows:

Denoting $\sqrt{(-1)}$ by i, let

$$\nu = n - n',$$

$$\mathrm{m} = \frac{n'}{\nu}, \qquad \kappa = \frac{\mu}{\nu^2};$$

also let

$$\tau = \nu(t - t_0), \qquad \zeta = e^{i\tau},$$

and put

$$u = x + iy, \qquad s = x - iy,$$

$$\mathrm{D} = -i\frac{d}{d\tau} = \zeta\frac{d}{d\zeta};$$

so that

$$\mathrm{D}\zeta^p = p\zeta^p;$$

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then u and s satisfy the differential equations :

$$\left\{D^2 + 2\mathrm{m}D + \tfrac{3}{2}\mathrm{m}^2 - \frac{\kappa}{(us)^{\frac{3}{2}}}\right\}u + \tfrac{3}{2}\mathrm{m}^2 s = 0,$$

$$\left\{D^2 - 2\mathrm{m}D + \tfrac{3}{2}\mathrm{m}^2 - \frac{\kappa}{(us)^{\frac{3}{2}}}\right\}s + \tfrac{3}{2}\mathrm{m}^2 u = 0,$$

each of which may of course be derived from the other by changing the sign of i.

These equations determine rigorously all the parts of the lunar co-ordinates which depend only on the ratio of the mean motions of the Sun and Moon and the lunar eccentricity.

Now let

$$u = \Sigma \mathrm{a}_p \zeta^{2p+1},$$
$$s = \Sigma \mathrm{a}_{-p-1} \zeta^{2p+1},$$

p having all integral values from $+\infty$ to $-\infty$.

The most simple method of obtaining the coefficients a_p is by substituting these values of u and s in the differential equations and equating coefficients. The presence of the factor $\frac{\kappa}{(us)^{\frac{3}{2}}}$ is a great additional source of complication in carrying out this process, and it is therefore desirable to eliminate it. This Mr. Hill effects by means of a first integral of the differential equations, and he at length obtains, as the final form of the equations to be employed,

$$D^2(us) - Du \,.\, Ds - 2\mathrm{m}(uDs - sDu) + \tfrac{9}{4}\mathrm{m}^2(u+s)^2 = C,$$
$$D(uDs - sDu - 2\mathrm{m}us) + \tfrac{3}{2}\mathrm{m}^2(u^2 - s^2) = 0.$$

By substituting the values of u and s in these differential equations, a system of algebraical equations connecting the a's is obtained from which they may be determined either numerically or algebraically in terms of m.

It will be noticed that the differential equations in this final form are such that in no case is it necessary to multiply together more than two infinite series.

The system of algebraical equations obtained by equating coefficients is infinite in number, and each equation contains an infinite number of terms involving the products of two a's.

It is found that a_p is of the $2p^{\text{th}}$ order with respect to m ; and, in solving the equations, only the terms of lowest order are in the first instance retained. The equations thus curtailed give the first four terms of the expansions in powers of m of a_1, a_{-1}, a_2, a_{-2}, &c. Thus a_1 and a_{-1} are affected with errors of the sixth order ; a_2 and a_{-2} with errors of the eighth order, and so on.

One of the great advantages of the method consists in the ease and rapidity with which the approximations can be extended ; for we have only to return to the original curtailed equations and augment each of them by the terms necessary to carry the

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approximation four orders higher; we then substitute in the new terms the values obtained from the first approximation, and in the old we determine what changes are produced by employing the more exact values instead of the first approximations. A repetition of this process carries the approximation four orders higher still, and so on.

Mr. Hill gives in full the equations which determine the a's correctly to quantities of the thirteenth order inclusive;* and he deduces from them the expansions of

$$\frac{a_{\pm 1}}{a_0}, \dots \frac{a_{\pm i}}{a_0}.$$

in powers of m as far as m^9.

From these expansions the values of $r \cos v$ and $r \sin v$, as series proceeding by cosines and sines of multiples of 2τ, are easily deduced by means of the equations

$$r \cos v = \Sigma a_p \cos 2p\tau$$

$$r \sin v = \Sigma a_p \sin 2p\tau.$$

the summations extending to infinity both ways. The coefficients in these series are given as far as the ninth order inclusive in powers of $\mathfrak{m}$, where

$$\mathfrak{m} = \frac{m}{m - \frac{1}{3}m}.$$

Not only does Mr. Hill give these coefficients in a literal form, which he afterwards reduces to numbers, but he gives also the direct numerical calculation, employing numerical values from the outset in the equations of condition. This latter method is found to be, as we should suppose, far less laborious than the literal development of the coefficients in powers of a parameter.

The series given by previous mathematicians are unsuitable for the determination of the motion of satellites having long periods of revolution about their primaries; and Mr. Hill is accordingly led to apply his method, which is available in such cases, to the exact determination of the expressions for $r \cos v$ and $r \sin v$ in terms of τ when $m = \frac{1}{10}, \frac{1}{9}, \frac{1}{8}, \dots \frac{1}{3}$. For moons of much longer lunations the method is not practicable, and it is necessary to resort to mechanical quadratures. The motion is

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* The final differential equations only suffice to determine the ratios of the a's to any one of themselves, as a_0. For the actual determination of a_0 in terms of μ and n recourse must be had to one of the original differential equations, and in this way the value of a_0, viz.

$$a_0 = \left(\frac{\mu}{n^2}\right)^{\frac{1}{3}} \left[1 - \tfrac{1}{6}m^2 + \tfrac{1}{3}m^3 + \tfrac{407}{2304}m^4 - \&c. \right]$$

is determined as far as the term involving m^9 inclusive. It is from the values of the coefficients a_p, which depend upon m alone, that the values of Θ_0, Θ_1, Θ_2, &c., which were required in finding the motion of the perigee, were derived by Mr. Hill.

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determined in this manner in the cases of $m = 2$ and $m = 1{\cdot}78265$. The latter value of m corresponds to a lunar orbit (relative to the moving axes) having cusps at the points of quadratures.

Brief as has necessarily been the preceding account of the mathematical processes employed by Mr. Hill, I hope that it has been sufficient to give an idea of their character. It is not often that the points of novelty in an application of mathematics to a well-known problem in one of the older sciences are so distinctive as to admit of explanation within a moderate compass. The present investigation is so exceptional in this respect that I have ventured to reproduce the main lines of the mathematical procedure.

Two obvious deviations from established custom at once attract attention: first, the use of rectangular instead of polar coordinates; and secondly, the introduction of the imaginary quantity i.

With regard to the choice of coordinates, it is known that in the elliptic theory the rectangular coordinates of a planet relative to a central body can be developed in series of sines and cosines, in which the coefficients are Besselian functions, although the longitude and latitude are far from having such simple expressions, and in fact are not even finitely expressible in terms of Besselian functions. If this be true in the elliptic theory, how much more likely, Mr. Hill asks, is a similar result to be true when the complexity of the problem is increased by the consideration of the disturbing forces? Thus there is reason to believe that the coefficients of the periodic terms in the developments of the rectangular coordinates are less complex functions of their parameters than in the case of polar coordinates. Another important advantage is afforded by the fact that when expressed in rectangular coordinates the differential equations assume purely algebraic forms, while in polar coordinates they involve circular functions.

The quantity i is used in order to avoid the multiplications of series of sines and cosines, and to reduce all the quantities to an algebraic form. It is at once apparent that the choice of $e^{\tau i}$ or ζ as independent variable greatly simplifies the expression of the series and the analysis in general. Having had occasion myself, in a very different branch of mathematics, to multiply together infinite series of sines and cosines, I have found the advantage of making a similar substitution, and I can bear testimony to the analytical simplicity and regularity produced in such processes by the substitution of

$$\Sigma_{-\infty}^{\infty} \mathrm{A}_p z^p \quad \text{and} \quad \frac{1}{i} \Sigma_{-\infty}^{\infty} \mathrm{B}_p z^p$$

for

$$\mathrm{A}_0 + 2\Sigma_1^{\infty} \mathrm{A}_p \cos 2p\theta \quad \text{and} \quad 2\Sigma_1^{\infty} \mathrm{B}_p \sin 2p\theta$$

A_{-p} and B_{-p} being supposed equal to A_p and $-\mathrm{B}_p$ respectively.

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It may be remarked that the vector u or $x+iy$ is a symbol to which a clear conception is now attached, and also that in the equation

$$x+iy=\Sigma \mathrm{a}_p \zeta^{2p+1}$$

the quantities a_0, a_1, a_{-1}, &c. are the radii of actual epicycles. The geometrical picture which this equation represents is that of a pure epicyclic system.

Mr. Hill employs the parameter $\mathfrak{m}$, which is equal to $\dfrac{m}{1-m}$, in place of m. Although $\mathfrak{m}$ is slightly larger than m it is found that in the series proceeding by powers of $\mathfrak{m}$ the coefficients are much smaller than in the case of those proceeding by powers of m, and that the series converge more rapidly. The quantity $\mathfrak{m}$, which has been mentioned above, was specially determined in order to render the series giving the a's as convergent as possible.

The superiority of the expansions in powers of $\mathfrak{m}$, both in convergence and simplicity of coefficients, is evident by comparing the values of $\dfrac{\mathrm{a}_1+\mathrm{a}_{-1}}{\mathrm{a}_0}$ and $\dfrac{\mathrm{a}_1-\mathrm{a}_{-1}}{\mathrm{a}_0}$, which when expressed in powers of $\mathfrak{m}$ assume the forms

$$-\{\mathfrak{m}^2+\tfrac{1}{2}\mathfrak{m}^3-\tfrac{2}{9}\mathfrak{m}^4+\tfrac{1}{36}\mathfrak{m}^5-\&c.\}$$

and

$$\tfrac{11}{8}\mathfrak{m}^2+\tfrac{5}{4}\mathfrak{m}^3+\tfrac{5}{72}\mathfrak{m}^4-\tfrac{11}{36}\mathfrak{m}^5+\&c.$$

respectively; the corresponding series, as given by Plana and Delaunay, being

$$-\{m^2+\tfrac{19}{6}m^3+\tfrac{131}{18}m^4+\tfrac{383}{27}m^5+\&c.\}$$

and

$$\tfrac{11}{8}m^2+\tfrac{59}{12}m^3+\tfrac{893}{72}m^4+\tfrac{2855}{108}m^5+\&c.$$

With regard to the general question of parameters, it is pointed out that the large numerical coefficients which occur in Delaunay's expansions are probably due to the choice of m as parameter, and that they might be very much diminished if a more suitable parameter were employed. Further, it is not necessary to use the same parameter throughout, and it may well happen that different parameters may be advantageously employed in the investigation of different classes of inequalities. Examples of the extraordinary increase of convergence which can be affected by a change of parameter are afforded by logarithmic series, and, in a much more striking form, by the various expansions of the elliptic functions. Mr. Hill, however, gives it as the result of his experience that no useful results are obtained by experimenting with the present known developments, and that in every case the proper parameter must be derived from *à priori* considerations suggested by the course of the integration.

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Passing now to general considerations, it will be noticed that Mr. Hill obtains the integrals of his differential equations by the method of indeterminate coefficients. This is the method which would seem to offer the best chance of discovering the law of the series. A further advantage of this method, which is not possessed by Delaunay's, is that it affords either the numerical values of the coefficients or the literal development in powers of a parameter.

Of all the general methods of treating the Lunar Theory as a whole, Delaunay's is by far the most elegant and complete, and is likely to become the classic method in this subject. When, however, the object is to investigate only a certain class of inequalities, it is not without its disadvantages; for example, even if we only desire to obtain the inequalities whose coefficients depend solely upon m, we are nevertheless obliged to develop the disturbing function R to all powers of e. Again, in Delaunay's method, the number of variables is doubled in order that all the differential equations may be of the first order; but when the coefficients are determined by direct substitution in the differential equations their order is immaterial. The two main characteristics which distinguish Mr. Hill's general analytical treatment of the lunar theory are the expression of the equations in terms of the coordinates instead of in terms of the elements, and the diminution of the number of variables and differential equations at the expense of increasing the order of the latter.

Immediately after the publication of Mr. Hill's first memoir (on the motion of the Moon's perigee), Professor Adams communicated a paper to the *Monthly Notices* (vol. xxxviii., p. 43), in which, after bearing testimony to the high merits of Mr. Hill's investigation, he mentions that he has been led to dwell thus particularly on this subject because his own researches in the lunar theory have followed in some respects a parallel course. Professor Adams states that he has long been convinced that the most advantageous way of treating the lunar theory is, first, to determine with all desirable accuracy the inequalities which are independent of e, e' and γ, and then in succession to find the inequalities which are of one dimension, two dimensions, and so on with respect to those quantities. Thus, the coefficient of any inequality in the Moon's coordinates would be represented by a series arranged in powers and products of e, e' and γ; and each term in this series would involve a numerical coefficient which is a function of m alone, and which admits of calculation for any given value of m without the necessity of developing it in powers of m. This method is particularly advantageous when we wish to compare our results with those of an analytical lunar theory such as Delaunay's, in which the eccentricities and the inclination are left indeterminate, since each numerical coefficient could be compared separately with its analytical development in powers of m. As it is only the series proceeding by powers of m in

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Delaunay's theory which have a slow rate of convergence, it is probable that all the sensible corrections required by Delaunay's coefficients would be found among the terms of low order in e, e' and γ.

Professor Adams proceeds: "The differential equations which would require solution in these successive operations after the determination of the inequalities independent of eccentricities and inclination would be all linear and of the same form.

"It is many years since I obtained the values of these last-named inequalities to a great degree of approximation, the coefficients of the longitude expressed in circular measure, and those of the reciprocal of the radius vector, or of the logarithm of the radius vector, being found to ten or eleven places of decimals.

"In the next place, I proceeded to consider the inequalities of latitude, or rather the disturbed value of the Moon's co-ordinate perpendicular to the ecliptic, omitting the eccentricities as before, and taking account only of the first power of γ.

"In this case the differential equation for finding z presents itself naturally in the form to which Mr. Hill reduces, with so much skill, the equations depending on the first power of the eccentricity of the Moon's orbit.

"In solving this equation I fell upon the same infinite determinant as that considered by Mr. Hill, and I developed it in a similar manner in a series of powers and products of small quantities, the coefficient of each such term being given in a finite form.

"The terms of the fourth order in the determinant were thus obtained by me on December 26, 1868. I then laid aside the further investigation of this subject for a considerable time, but resumed it in 1874 and 1875; and on December 2 in the latter year I carried the approximation to the value of the determinant as far as terms of the twelfth order, or to the same extent as that which has been attained by Mr. Hill. I have also succeeded in reducing the determination of the inequalities of longitude and radius vector which involve the first power of the lunar eccentricity to the solution of a differential equation of the second order; but my method is much less elegant than that of Mr. Hill.

"Immediately after Mr. Hill's paper reached me I wrote to him expressing my opinion of its merits, and telling him what I had done in the same direction; and I received from him a very cordial and friendly letter in reply.

"The equation which I had obtained by equating the above-mentioned determinant to zero differed in form from Mr. Hill's; and on making the reductions required to make the two results immediately comparable, I found that there was an agreement between them except in one term of the twelfth order. On examining my work I found that this arose from a simple error of transcription in a portion of my work, and that when this had

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been rectified my result was in entire accordance with Mr. Hill's. The calculations by which I have found the value of the determinant are very different in detail from those required by Mr. Hill's method, and appear to be considerably more laborious."

The differential equation which determines z, the Moon's coordinate perpendicular to the equator, is

$$\frac{d^2z}{dt^2}+\left(\frac{\mu}{r^3}+\frac{m'}{r_1{}^3}\right)z=0.$$

This equation thus presents itself originally (as mentioned by Professor Adams) in the same form,

$$\frac{d^2\mathrm{w}}{d\tau^2}+\Theta\mathrm{w}=0,$$

as that to which Mr. Hill succeeds in reducing the equations upon which the inequalities involving the first power of the lunar eccentricity depend.

Professor Adams puts

$$\frac{\mu}{r^3}+\frac{m'}{r_1{}^3}=(n-n')^2\left\{q^2+2\Sigma_1^\infty q_p\cos 2p(n-n')t\right\}$$

and Mr. Hill puts

$$\Theta=\Theta_0+\Theta_1\cos 2\tau+\Theta_2\cos 4\tau+\&c.$$

The same determinant is, therefore, reached in both cases, and there is no great difference in the notation. Professor Adams's form of the developed value of the determinant is given on p. 47 of volume of the *Monthly Notices* referred to. By means of this developed expression, the value of g, which stands to the node in the same relation that c does to the perigee, is found to fifteen places of decimals. I may mention that in this paper Professor Adams also gives the numerical values of the terms in the expansions of $g-1$ and $c-1$ both in powers of m and of m. In each case the development in powers of m is much the more advantageous; and he states that he has found that a similar advantage results from the employment of m instead of m in the development of the Moon's periodic inequalities.

Before considering in a more general light the inferences that we may draw from the similarity of the researches of Professor Adams and Mr. Hill, I should like to draw attention to the fundamental character of Mr. Hill's infinite determinant in the lunar theory, and possibly also in other applications of mathematics.

The differential equation

$$\frac{d^2\mathrm{w}}{d\tau^2}+\Theta\mathrm{w}=0.$$

may be regarded as a canonical form.* Mr. Hill was led to it by

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* Riccati's equation and several other well-known equations are reducible to this form, which has been much studied; but, so far as I know, the case in which Θ is a periodic series has not been considered.

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considering the motion of the perigee, Professor Adams by considering the motion of the node. It is probable that many other inequalities can be made to depend upon the solution of an equation of this form. The form of Θ, viz.,

$$\Theta_0 + \Theta_1 \cos 2\tau - \Theta_2 \cos 4\tau + \&c.,$$

where Θ_p is of the $2p$th order, is very general; and so also is the method of solution by indeterminate coefficients.

The determinant in question, therefore, possesses considerable mathematical interest of its own, and its properties and development seem likely to admit of a wide application in mathematical researches in which periodic functions are involved.

The fact that the work carried out independently by Professor Adams and Mr. Hill should have run so much upon the same lines, and have so much actually in common, affords strong evidence that it is in this direction that we must look, in the present condition of the lunar theory, for any further important advances. It is almost impossible to study the writings to which I have referred without becoming impressed by this feeling. Mr. Hill puts the matter very clearly when he points out that the great majority of the mathematicians who have worked at the lunar theory have had before them, as their ultimate aim, the construction of tables; that is to say, they have viewed the problem from the standpoint of practical astronomy rather than of mathematics. This accounts for the restricted choice of variables and parameters. "Again," proceeds Mr. Hill, "their object compelling them to go over the whole field, they have neglected to notice many minor points of great interest to the mathematician, simply because the knowledge of them was unnecessary for the formation of tables. But the developments having now been carried extremely far, without completely satisfying all desires, one is led to ask whether such modifications cannot be made in the processes of integration, and such coordinates and parameters adopted, that a much nearer approach may be had to the law of the series, and at the same time their convergence augmented."

In recent years it has come to be generally believed that a worker had but little chance of performing useful service in the lunar theory unless he was prepared to make it the study of his life. The belief also has been prevalent that the mathematical portion of the treatment of the subject has been worked out, and that there was no scope for the display of mathematical skill or the employment of modern mathematical methods. Until some great discovery should change the face of the whole subject, it has seemed likely that patience and diligence in traversing with greater care the old lines, and extending still further developments already carried to a wonderful extent, would be all that was required to perfect the theory. Considering, on the other hand, the attractiveness of the new and rapidly progressing

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branches of pure mathematics, and of many recent applications of mathematics to physical science, it is scarcely to be wondered at that so few of the younger generation of mathematicians should have included the lunar theory within their subjects of research. The papers of Mr. Hill's which I have described, and certain recent papers of Professor Adams's, have invested the lunar theory with a new mathematical interest, and have shown that in the treatment of the special problems included in the subject there is an ample opportunity not only for the application of existing mathematical methods, but even for the discovery of new ones. These papers show also that it is possible for the mathematician to confine himself to these special problems without attempting to cover the whole ground of the lunar theory. I hope that this is the dawn of a new day in the history of the lunar problem, and that, now that the whole territory has been mapped out by Plana and Delaunay, it will be found that the special investigations offer a tempting field to the mathematician. So far from the subject having been exhausted by the general methods which have been applied to it as a whole, I believe that the future will show that they have but cleared the ground and disclosed to view the objects to which mathematical investigation may with the greatest advantage be directed.

I have now described, though necessarily in a very imperfect manner, Mr. Hill's mathematical researches upon special problems in the lunar theory, the originality and peculiar value of which the Council have been especially desirous of recognising by the award of their medal. I can refer but briefly to Mr. Hill's other important contributions to physical astronomy, most of which, however, have been already noticed in our Annual Reports.

The American *Journal* contains two other papers of his. The first of these, in the fourth volume, is a short note on "Hansen's General Formulæ for Perturbations," in which an equation of Hansen's is expressed in a form which is not only more simple, but also more convenient for computation. The second, in the sixth volume, is entitled "On Certain Possible Abbreviations in the Computation of the Long-Period Inequalities of the Moon's Motion, due to the Direct Action of the Planets." Hansen has characterised the calculation of the coefficients of these inequalities as extremely difficult; but Mr. Hill finds that if the shortest methods are followed this is no longer the case. In this characteristic paper, Mr. Hill, following the principles which, as we have seen, have guided him in all his work, attacks the question from the very beginning by a direct analytical investigation. He thus obtains finally, not only a new method for the actual computation of the inequalities, but also some results of more purely mathematical interest.

Three elaborate memoirs have been published by Mr. Hill among the astronomical papers prepared for the use of the *American Ephemeris and Nautical Almanac* (i) "On Gauss's

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Method of Computing Secular Perturbations, with an Application to the Action of *Venus* on *Mercury*" (1881); (ii) "Determination of the Inequalities of the Moon's Motion, which are produced by the Figure of the Earth: a Supplement to Delaunay's Lunar Theory" (1884); (iii) "On Certain Lunar Inequalities, due to the Action of *Jupiter*, and discovered by Mr. E. Neison" (1885).

The object of the first of these papers is to reproduce in a form practically useful for astronomers the results given by Gauss in his celebrated memoir, "Determinatio Attractionis &c." In the original memoir the method is not presented in such a manner that it admits of easy application to an actual case. In this paper Mr. Hill places Gauss's process within the reach of any computer possessed of ordinary mathematical knowledge. Gauss's investigation is reproduced in full, with the addition of the details necessary for its application; and, finally, the formulæ themselves, in the shape in which they are required by the practical calculator, as well as the necessary tables of elliptic integrals, are given. As an example, the computation of the secular perturbations of *Mercury*, produced by the action of *Venus*, are worked out; the final results agreeing very closely with those of Leverrier. It is to be hoped that this paper may have the effect of bringing into more general practical use Gauss's very beautiful and perfect method.

The second memoir is described by the author in its title as a supplement to Delaunay's Lunar Theory At the time of Delaunay's death the inequalities caused by the disturbing action of the Sun were complete; but the small effects produced by the spheroidal figure of the Earth, and other minor portions of the subject, were untouched. Mr. Hill's object in this memoir is to supplement Delaunay's great work by determining, in a literal form, all the inequalities of the Moon which arise from the figure of the Earth to the same degree of algebraical approximation as that adopted by Delaunay in determining the solar perturbations—that is, to terms of the seventh order inclusive. The value of a constant which enters into the coefficients and depends upon the figure of the Earth is determined by an elaborate discussion of numerous pendulum experiments; and by the substitution of the values of m, e, e', γ and this constant, the numerical values of the coefficients are deduced. The whole of this elaborate investigation seems to have been carried out with great skill and care. I should mention that, in his treatment of the subject, Mr. Hill adopts in principle the usual method of applying Delaunay's theory to the determination of the lunar inequalities which are due to causes other than the Sun's action, and that there is possibly a slight imperfection in this method. It is probable, however, that the differences due to this cause are insensible.

In the third of the memoirs referred to Mr. Hill effects the determination of the coefficients of two lunar inequalities,

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due to the action of *Jupiter*, to which Mr. Neison drew attention in vol. xxxvii. of the *Monthly Notices.* Mr. Hill's values of these coefficients do not agree with those assigned to them by Mr. Neison. In the case of the second inequality Mr. Hill's value amounts to only about one-tenth of Mr. Neison's.

It is nine years since Mr. Hill undertook for the office of the *American Ephemeris and Nautical Almanac* an investigation of the motions of *Jupiter* and *Saturn*, with the view of constructing tables of these two planets. From its commencement this great work has occupied nearly the whole of his time; and, even if no hindrance occurs, it must continue to do so for the next three or four years. It is well known that the theory of *Saturn* is in an unsatisfactory state, Leverrier's tables failing to represent adequately the observations. A special interest therefore attaches to these new investigations which have been carried out by means of Hansen's very refined methods. The work has so far progressed very satisfactorily. The final expressions for the coordinates of the two bodies, together with the details of a preliminary comparison of them with observations made for the purpose of ascertaining what corrections the perturbations might need on account of errors in the provisionally assumed elements, were published last year in the "Astronomische Nachrichten" (Nos. 2,705–2,706). When these investigations, in which the approximations are carried to a much greater extent than in any previous treatment of the subject, are completed, the whole will form by far the largest and most complete investigation of the kind that has yet been performed on the American continent.

It is doubtless due to the engrossing character of this work that only two chapters of the "Researches on the Lunar Theory" were published. There is abundant evidence that Mr. Hill's researches must have been extended far beyond those given in these two chapters at the time when his new labours deprived him of the leisure necessary for completing and preparing them for press; and I trust that when the Theory and Tables of *Jupiter* and *Saturn* shall have been published, Mr. Hill may be enabled to resume his mathematical researches at the point at which their publication ceased, and to carry out his original intention to its full extent.

The President then, delivering the Medal to the Foreign Secretary, addressed him in the following terms:

Dr. Huggins,—May I ask you in transmitting this medal to Mr. Hill to assure him of our high appreciation of the value of his contributions to physical astronomy, and to express to him our earnest hope that his life may long be spared, and that he may have health and strength to complete the splendid work upon which he is now engaged, and to enrich our science still further by his mathematical researches.